Category Theory

Remaining TODOs: 37

This document contains notes for, and (not necessarily correct) solutions for exercises from Mac Lane, S. (1997), *Categories For the Working Mathematician* (2nd ed.).

Contents

I Categories, functors, and natural transformations

I.1 Axioms for categories

No exercises

I.2 Categories

Definition I.2.1

A directed graph is a set O of objects, a set A of arrows, and two functions $\operatorname{dom}, \operatorname{cod} : A \to O$.

Definition I.2.2

In a directed graph, the set of composable pairs of arrows is the set

$$A \times_O A := \{\langle g, f\rangle \mid g, f \in A, \operatorname{dom} g = \operatorname{cod} f\}.$$

Definition I.2.3

A category is a directed graph with two addition functions, $\mathrm{id} : O \to A; c \mapsto \mathrm{id}_c$ and $\circ : A \times_O A \to A; \langle g, f\rangle \mapsto g \circ f$ such that

$$\operatorname{dom}(\mathrm{id}\, a) = a = \operatorname{cod}(\mathrm{id}\, a), \quad \operatorname{dom}(g \circ f) = \operatorname{dom} f, \quad \operatorname{cod}(g \circ f) = \operatorname{cod}(g)$$

for all objects $a \in O$ and all composable pairs of arrows $\langle g, f\rangle \in A \times_O A$, and such that the associativity and unit axioms hold.

For a category $\mathcal{C}$, we write $\operatorname{Ob}(\mathcal{C}) = O, \operatorname{Ar}(\mathcal{C}) = A$, and $c \in \mathcal{C}, f \in \mathcal{C}$ to mean $c \in \operatorname{Ob}(\mathcal{C}), f \in \operatorname{Ar}(\mathcal{C})$ respectively.

Definition I.2.4

Given a category $\mathcal{C}$ and two objects $b, c \in \mathcal{C}$, the hom-set

$$\hom_{\mathcal{C}}(b, c) := \{f \mid f \in \mathcal{C}, \operatorname{dom} f = b, \operatorname{cod} f = c\}$$

is the set of arrows from b to c in $\mathcal{C}$.

Example I.2.5

0 is the empty category.
1 is the category with one object and one (identity) arrow.
2 is the category with two objects a, b, and one arrow $f : a \to b$ (plus the identity arrows)
3 is the category with three objects a, b, c with non-identity arrows from $f : a \to b, g : b \to c$ and $g \circ f : a \to c$.

Definition I.2.6

A category is discrete if every arrow is an identity.

Definition I.2.7

A monoid is a category with just one object.

Remark I.2.8

Arrows can be parallel - that is, given objects a, b, there can be several distinct arrows between a and b.

Example I.2.9

A group is a monoid in which every arrow has a two-sided inverse under composition.

Definition I.2.10

For a commutative ring K, the category $\mathbf{Matr}_K$ has objects $\mathbb{N}_0$ and $\hom(n, m) = \mathcal{M}_{m\times n}(K)$, with composition the usual matrix product.

Definition I.2.11

For V a family of sets, the category $\mathbf{Ens}_V$ has object set V, and arrows all functions between them, with composition the usual function composition.

Definition I.2.12

A preorder is a category in which, for each pair of objects p, p', there is at most one arrow $p \to p'$.

We define a binary operation $\leq$ on a preorder $\mathcal{P}$ such that $p \leq p'$ iff there is an arrow $p \to p'$ in $\mathcal{P}$.

Remark I.2.13

Partial orders are preorder with antisymetry.

TODO more examples?

I.3 Functors

Definition I.3.1

For categories $\mathcal{B}, \mathcal{C}$, a functor $T : \mathcal{B} \to \mathcal{C}$ consists of an object function which assigns to each object $b \in \mathcal{B}$ an object $Tb \in \mathcal{C}$, and an arrow function which assigns to each arrow $f \in \mathcal{B}$ an arrow $Tf \in \mathcal{C}$, such that the functor axioms

$$T(1_c) = 1_{Tc}, \qquad T(g \circ f) = Tg \circ Tf$$

are satisfied.

Example I.3.2

The power set functor $\mathscr{P} : \mathbf{Set} \to \mathbf{Set}$ assigns to each object X the usual power set $\mathscr{P}X$, and sends each arrow $f : X \to Y$ to $\mathscr{P}f : \mathscr{P}X \to \mathscr{P}Y$ such that, for each $S \subseteq X$, $(\mathscr{P}f)S = fS$.

Remark I.3.3

Functors can be composed by composing object and arrow functions respectively.

Remark I.3.4

For each category $\mathcal{C}$ there is an identity functor $I_{\mathcal{C}}$ which acts as an identity under functor composition.

Definition I.3.5

An isomorphism between categories $\mathcal{B}, \mathcal{C}$ is a functor $T : \mathcal{B} \to \mathcal{C}$ which is a bijection. (Equivalently, T has a two-sided sided inverse).

Definition I.3.6

A forgetful functor is one that simply "forgets" some or all of the structure of an algebraic object.

Definition I.3.7

A functor $T : \mathcal{B} \to \mathcal{C}$ is full when, for each $b, b' \in \mathcal{B}$, $g : Tb \to Tb' \in \mathcal{C}$, there is an arrow $f : b \to b' \in \mathcal{B}$ such that $Tf = g$.

That is, for each pair of objects c, c' in the image of T, every arrow between c and c' is also in the image of T.

Definition I.3.8

A functor $T : \mathcal{B} \to \mathcal{C}$ is faithful, or an embedding, when for each $b, b' \in \mathcal{B}$ and parallel arrows $f_1, f_2 : b \to b'$, $Tf_1 = Tf_2$ implies $f_1 = f_2$.

Exercise I.3.1

TODO quotients, Lie algebras

Exercise I.3.2

TODO

Exercise I.3.3

TODO functor interpretation in special categories

Exercise I.3.4

TODO prove no functor sending groups to centres

Exercise I.3.5

TODO find two different functors on Grp with object function the identity

I.4 Natural transformations

Definition I.4.1

Given categories $\mathcal{B}, \mathcal{C}$ and functors $S, T : \mathcal{B} \to \mathcal{C}$, a natural transformation $\tau : S \dot{\to} T$ is a function that assigns to each object $b \in \mathcal{B}$ an arrow $\tau_b \in \mathcal{C}$ from $Tb \to Sb$ such that, for any objects $b, b' \in \mathcal{B}$ and arrow $f : b \to b'$, $Tf \circ \tau_b = \tau_{b'} \circ Sf$. That is, for any such b, b', f, the following diagram commutes:

$$\begin{tikzcd}
Sb \arrow[r, "\tau_b"] \arrow[d, "Sf"'] & Tb \arrow[d, "Tf"] \\
Sb' \arrow[r, "\tau_{b'}"] & Tb'
\end{tikzcd}$$

Remark I.4.2

Despite the notation, a natural transformation isn't a morphism between functors. It is an assignment from objects to (preexisting!) arrows - and the arrows are 'natural' in the sense that they are just arrows that would be pointing from Sb to Tb anyway. The only 'interesting' part of a natural transformation is choosing which arrows to assign in the case of there being parallel arrows, in order to make each square as above commute.

Example I.4.3

$$\begin{tikzcd}
Sa \arrow[rr, "\tau a"] \arrow[dr, "Sf"] \arrow[dd, "Sh"'] & & Ta \arrow[dr, "Tf"] \arrow[dd] & \\
& Sb \arrow[rr, "\tau b"] \arrow[dl, "Sg"'] & & Tb \arrow[dl, "Tg"'] \\
Sc \arrow[rr, "\tau c"] & & Tc &
\end{tikzcd}$$

Figure I.1: An example of a natural transformation, $\tau : S \dot{\to} T$ for functors $S, T : \mathbf{3} \to \mathcal{C}$

Definition I.4.4

A natural isomorphism is a natural transformation for which each τ_b is invertible in $\mathcal{C}$. Then we write $\tau : S \cong T$.

Definition I.4.5

An equivalence between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $S : \mathcal{C} \to \mathcal{D}, T : \mathcal{D} \to \mathcal{C}$ such that $I_{\mathcal{C}} \cong T \circ S$ and $I_{\mathcal{D}} \cong S \circ T$.

Exercise I.4.1

Let S be a fixed set, and X^S the set of all functions $h : S \to X$. Show that $X \mapsto X^S$ is the object function of a functor $\mathbf{Set} \to \mathbf{Set}$, and that evaluation $e_X : X^S \times S \dot{\to} X$, defined by $e(h, s) = h(s)$, the value of the function h at $s \in S$, is a natural transformation.

Solution. Denote such a functor $T : \mathbf{Set} \to \mathbf{Set}$. Let the arrow function of T be $T\,(f : X \to X') : X^S \to X'^S := f \circ -$ defined by $(f \circ -)\, h = f \circ h$.

We can verify that this satisfies the functor axioms:

$$\begin{aligned} T(1_X) &= 1_X \circ - \\ &= 1_{X^S} \end{aligned}$$

and, for all $h \in X^S$,

$$\begin{aligned} T(f) \circ T(g)\, h &= ((f \circ -) \circ (g \circ -))\, h \\ &= (f \circ -)\,(g \circ h) \\ &= f \circ (g \circ h) \\ &= (f \circ g) \circ h \\ &= ((f \circ g) \circ -)\, h \\ &= T(f \circ g)\, h \end{aligned}$$

so $T(f \circ g) = T(f) \circ T(g)$.

Then note that

$$\begin{aligned}(f \circ e_X)\,(h, s) &= f(h(s)) \\ &= (f \circ h)\; s \\ &= e_{X'}(f \circ h, s) \\ &= (e_{X'} \circ (f \circ -))\,(h, s) = (e_{X'} \circ (T(f) \times 1_S))\,(h, s),\end{aligned}$$

i.e. $f \circ e_X = e_{X'} \circ (T(f) \times 1_S)$.

Therefore the following diagram commutes for any $f : X \to X'$:

$$\begin{tikzcd} X^S \times S \arrow[r, "e_X"] \arrow[d, "T(f) \times 1_S"'] & X \arrow[d, "f"] \\ X'^S \times S \arrow[r, "e_{X'}"] & X' \end{tikzcd}$$

so e is a natural transformation that assigns to each set X an arrow e_X.

Exercise I.4.2

Show that, if H is a fixed group, then $G \mapsto H \times G$ defines a functor $H \times - : \mathbf{Grp} \to \mathbf{Grp}$, and each morphism $f : H \to K$ of groups defines a natural transformation $H \times - \stackrel{\cdot}{\to} K \times -$.

Solution.

For a fixed group H, define the functor $H \times - : \mathbf{Grp} \to \mathbf{Grp}$ by

$$(H \times -)\, G \mapsto H \times G$$

and

$$(H \times -)\, g \mapsto 1_H \times g$$

for any group homomorphism g.

We can verify that this definition satisfies the functor axioms:

$$\begin{aligned}(H \times -)\, 1_G &= 1_H \times 1_G \\ &= 1_{H \times G} \\ &= 1_{(H \times -)\, G}\end{aligned}$$

and

$$\begin{aligned}(H \times -)\,(g_1 \circ g_2) &= 1_H \times (g_1 \circ g_2) \\ &= (1_H \times g_1) \circ (1_H \times g_2) \\ &= ((H \times -)\, g_1) \circ ((H \times -)\, g_2).\end{aligned}$$

Then for any morphism $f : H \to K$ of groups, for any groups G_1, G_2, and for any group homomorphism $g : G_1 \to G_2$, the following diagram commutes:

$$\begin{tikzcd} H \times G_1 \arrow[r, "f_{G_1}"] \arrow[d, "g_H"'] & K \times G_1 \arrow[d, "g_K"'] \\ H \times G_2 \arrow[r, "f_{G_2}"] & K \times G_2 \end{tikzcd}$$

where $g_G : G \times G_1 \to G \times G_2 := 1_G \times g$ and $f_G : H \times G \to K \times G := f \times 1_G$.

This commutes because

$$\begin{aligned} g_K \circ f_{G_1} &= (1_K \times g) \circ \left(f \times 1_{G_1}\right) \\ &= (1_K \circ f) \times \left(g \circ 1_{G_1}\right) \\ &= (f \circ 1_H) \times \left(1_{G_2} \circ g\right) \\ &= f_{G_2} \times g_H. \end{aligned}$$

Hence f defines a natural transformation that assigns f_G as above for every object G.

Exercise I.4.3

If G and H are groups (regarded as categories with one object each), and $S, T : G \to H$ are functors (group homomorphisms), then there is a natural transformation $S \dot{\to} T$ iff S and T are conjugate, i.e. $\exists h \in H$ s.t, $Tg = h(Sg)h^{-1}$ for all $g \in G$.

Solution.

"$\Longrightarrow$":

Since there is a natural transformation $\tau : S \dot{\to} T$, for every $g \in G$, the following diagram commutes:

$$\begin{tikzcd} *_H \arrow[loop left, "Sg"] \arrow[r, "\tau_{*_G}"] & *_H \arrow[loop right, "Tg"] \end{tikzcd}$$

Figure I.4: The natural transformation $\tau : S \to T$

with $\tau_{*_G} =: h$ for $h \in H$.

Then for every $g \in G$,

$$\begin{aligned} &\tau_{*_G} \circ (Sg) = (Tg) \circ \tau_{*_G} \\ &\text{so } h(Sg) = (Tg)h \\ &\text{so } Tg = h(Sg)h^{-1}. \end{aligned}$$

"$\Longleftarrow$":

Suppose that there exists some $h \in H$ s.t. for all $g \in G$, $Tg = h(Sg)h^{-1}$.

Then $(Tg)h = h(Sg)$ for all $g \in G$, so Figure I.4 commutes for all $g \in G$. Hence the assignment $\tau_{*_G} = h$ defines a natural transformation $\tau : S \dot{\to} T$.

Exercise I.4.4

TODO (involves preorders)

Exercise I.4.5

Show that:

a) Every natural transformation $\tau : S \dot{\to} T$ defines a function (also called τ) which sends each arrow $f : c \to c'$ of $\mathcal{C}$ to an arrow $\tau f : Sc \to Tc'$ of $\mathcal{B}$ in such a way that $Tg \circ \tau f = \tau(g\, f) = \tau g \circ Sf$ for each composable pair $\langle g, f\rangle$.

b) Conversely, every such function τ comes from a unique natural transformation with $\tau_c = \tau(1_c)$.

Solution.

a) Given a natural transformation $\tau : S \dot{\to} T$, the following diagram commutes for each $f : c \to c'$ of C:

$$\begin{tikzcd} Sc \arrow[r, "\tau_c"] \arrow[d, "Sf"'] \arrow[dr, dashed, "\tau f"] & Tc \arrow[d, "Tf"] \\ Sc' \arrow[r, "\tau_{c'}"] & Tc' \end{tikzcd}$$

where $\tau f = \tau_{c'} \circ Sf = Tf \circ \tau_c$.

Then taking a composable pair $\langle g : c' \to c'', f : c \to c'\rangle$,

$$\begin{aligned} Tg \circ \tau f &= Tg \circ (Tf \circ \tau_c) \\ &= (Tg \circ Tf) \circ \tau_c \\ &= T(g \circ f) \circ \tau_c \\ &= \tau(gf) \\ &= \tau_{c''} \circ S(g \circ f) \\ &= \tau_{c''} \circ (Sg \circ Sf) \\ &= (\tau_{c''} \circ Sg) \circ Sf \\ &= \tau g \circ Sf. \end{aligned}$$

b) Consider such a function τ, then we construct a natural transformation τ, assigning $\tau_c := \tau(1_c)$, that defines that function.

Then for any arrow $f : c \to c'$,

$$\begin{aligned} Tf \circ \tau(1_c) &= \tau(f\,1_c) \\ \implies Tf \circ \tau_c &= \tau f \end{aligned}$$

and

$$\begin{aligned} \tau(1_c\,f) &= \tau(1_c) \circ Sf \\ \implies \tau f &= \tau_c \circ Sf, \end{aligned}$$

hence

$$Tf \circ \tau_c = \tau_c \circ Sf,$$

which satisfies naturality. Moreover, this natural transformation is unique because by the unique definition of τ_c.

Remark

This gives an "arrows only" description of a natural transformation.

Exercise I.4.6

Let F be a field. Then the category of all finite-dimensional vector spaces over F, with morphisms all linear transformations, $\mathbf{FinVect}_F$, is equivalent to the category $\mathbf{Matr}_F$ (recalling that $\mathbf{Matr}_F$ is the category with objects all positive integers, and arrows from n to m all $m \times n$ matrices under F).

Solution.

STP that there exist functors $S : \mathbf{Matr}_F \to \mathbf{FinVect}_F$ and $T : \mathbf{FinVect}_F \to \mathbf{Matr}_F$ and natural isomorphisms $I_{\mathbf{FinVect}_F} \cong T \circ S$, $I_{\mathbf{Matr}_F} \cong S \circ T$, with I_C the identity functor for a category C.

For each vector space V, fix a basis B_V. Define $T\,V := \dim V, T\,h := \mathrm{rep}_{B_V,B_V} h$ for each vector space V and linear transformation $h : V \to W$, and define $S\,n = F^n$, $S\,M = (\vec{v} \mapsto M\vec{v})$ for each $n \in \mathbb{N}$ and $M \in \mathcal{M}_{m\times n}$.

We can verify that S, T are functors:

$$\begin{gathered} T(1_V) = I = 1_n = 1_{TV}; \\ T(g \circ h) = \mathrm{rep}_{B_V,B_W}(g \circ h) = T\,g \circ T\,h; \\ S(1_n) = 1_{F^n}; \\ S(P \circ G) = \vec{v} \mapsto PQ\vec{v} = S\,P \circ S\,Q. \end{gathered}$$

Then for each vector space V define a vector isomorphism $\tau_V : V \to F^n$ defined by $V \mapsto (T \circ S)\,V$, and for each $n \in \mathbb{N}$ define $\sigma_n := \mathrm{id}$.

Then the following diagrams commute:

$$\begin{tikzcd} V \arrow[r, "\tau_V"] \arrow[d, "h"'] & (T \circ S)\, V \arrow[d, "(T \circ S)\, h"] \\ W \arrow[r, "\tau_W"] & (T \circ S)\, W \end{tikzcd} \qquad \begin{tikzcd} n \arrow[r, "\sigma_n"] \arrow[d, "M"'] & (S \circ T)\, n \arrow[d, "(S \circ T)\, M"] \\ m \arrow[r, "\sigma_m"] & (S \circ T)\, m \end{tikzcd}$$

hence $I \cong S \circ T$, $I \cong T \circ S$, so $\mathbf{Matr}_F$ and $\mathbf{FinVect}_F$ are equivalent.

I.5 Monics, epics, and zeroes

Definition I.5.1

An arrow $e : a \to b$ is invertible in $\mathcal{C}$ if there is an arrow $e^{-1} : a \to b \in \mathcal{C}$ such that $e \circ e^{-1} = 1_b$ and $e^{-1} \circ e = 1_a$.

Remark I.5.2

If such an e^{-1} exists, it is unique.

Definition I.5.3

Two objects $a, b \in \mathcal{C}$ are isomorphic, written $a \cong b$, if there is an invertible arrow (an isomorphism) between them.

Definition I.5.4

An arrow $m : a \to b$ is monic in $\mathcal{C}$ when, for any parallel arrows $f_1, f_2 : d \to a$, $m \circ f_1 = m \circ f_2$ implies $f_1 = f_2$.

That is, m is monic when it is left-cancellable.

Remark I.5.5

In **Set** and **Grp**, the monic arrows are precisely the injections/monomorphisms.

Definition I.5.6

An arrow $h : a \to b$ is epi in $\mathcal{C}$ when it is right-cancellable; that is, for parallel arrows $g_1, g_2 : d \to a$, $g_1 \circ h = g_2 \circ h$ implies $g_1 = g_2$.

Remark I.5.7

In **Set** the epi arrows are the surjections/epimorphisms.

Definition I.5.8

For an arrow $h : a \to b$, a section of h in $\mathcal{C}$ is a right-inverse of h in $\mathcal{C}$ (i.e. r is section of h if $h \circ r = 1_b$).

Proposition I.5.9

If h has a section, it is epi.

Proof. TODO (not given in MacLane) □

Remark I.5.10

The converse (that an epi arrow has a section) is true in **Set** but not **Grp**. (TODO: proof?)

Definition I.5.11

A retraction is a left-inverse.

Remark I.5.12

Sections and retractions are not necessarily unique.

Definition I.5.13

If we have arrows g, h with $g \circ h = 1$, then g is a split epi, and h a split monic.

Definition I.5.14

An arrow $f : b \to b$ is idempotent if $f \circ f = f$.

Proposition I.5.15

If $g \circ h = 1$, then $f = h \circ g$ is defined and is idempotent ($f \circ f = f$).

Definition I.5.16

We say that an idempotent f splits when there exists arrows g, h with $f = hg$ and $gh = 1$.

Definition I.5.17

An object t is terminal in $\mathcal{C}$ if, for every object object $a \in \mathcal{C}$, there is exactly one arrow $a \to t$ in $\mathcal{C}$.

Proposition I.5.18

Any two terminal objects in $\mathcal{C}$ are isomorphic.

Proof. TODO (not given) □

Definition I.5.19

An object s is initial in $\mathcal{C}$ if, for every object $a \in \mathcal{C}$, there is exactly one arrow $s \to a$ in $\mathcal{C}$.

Definition I.5.20

A null object in $\mathcal{C}$ is one that is both initial and terminal.

Proposition I.5.21

If $\mathcal{C}$ has a null object, then it is unique up to isomorphism.

Proof. TODO □

Proposition I.5.22

Given a null object $z \in \mathcal{C}$, for any objects $a, b \in \mathcal{C}$, there is a unique arrow $f = g \circ h$ for $g : a \to z$ and $h : z \to b$.

Proof. TODO (not given) □

Definition I.5.23

Such an arrow is called the zero arrow from a to b.

Proposition I.5.24

Any composite with a zero arrow is a zero arrow.

Proof. TODO (not given) □

Definition I.5.25

A groupoid is a category in which every arrow is invertible.

Remark I.5.26

Every object x in a groupoid $\mathcal{G}$ forms a group $\hom_{\mathcal{G}}(x, x)$.

If there is an arrow $f : x \to x'$, then $\hom_{\mathcal{G}}(x, x) \cong \hom_{\mathcal{G}}(x', x')$ under conjugation.

Definition I.5.27

A groupoid is connected if there is an arrow joining any two of its objects.

Remark I.5.28

A connected groupoid can be determined up to isomorphism by a group and the set of objects.

Exercise I.5.1

TODO find arrow epi and monic but not invertible

Exercise I.5.2

TODO prove monic compose monic is monic, same for epi

Exercise I.5.3

TODO If $g \circ f$ is monic, is g monic?

Exercise I.5.4

TODO rings

Exercise I.5.5

TODO show epi in Grp is surjection

Exercise I.5.6

TODO show that all idempotents split in **Set**

Exercise I.5.7

TODO

Exercise I.5.8

TODO

Exercise I.5.9

TODO show that if $T : C \to B$ is faithful and Tf monic, f monic

I.6 Foundations

TODO

I.7 Large Categories

TODO

I.8 Hom-Sets

TODO

II Constructions on Categories

II.1 Duality

TODO

II.2 Contravariance and Opposites

TODO

II.3 Products of Categories

Definition II.3.1

For categories $\mathcal{B}$, $\mathcal{C}$, the product category $\mathcal{B} \times \mathcal{C}$ has objects $\langle b, c \rangle$ for each object b of $\mathcal{B}$ and c of $\mathcal{C}$, and arrows $\langle f, g \rangle$ for each arrow f of $\mathcal{B}$ and g of $\mathcal{C}$, with the composite defined by

$$\langle f, g \rangle \circ \langle g', g' \rangle = \langle f \circ f', g \circ g' \rangle.$$

Definition II.3.2

The product category $\mathcal{B} \times \mathcal{C}$ is equipped with projection functors $P : \mathcal{B} \times \mathcal{C} \to \mathcal{B}, Q : \mathcal{B} \times \mathcal{C} \to \mathcal{C}$ defined by

$$P\langle f, g \rangle = f; \quad Q\langle f, g \rangle = g$$

for objects and arrows f of $\mathcal{B}$, g of $\mathcal{C}$.

Theorem II.3.3 (universal property of product categories)

For product category $\mathcal{B} \times \mathcal{C}$ and any category $\mathcal{D}$ with functors $R : \mathcal{D} \to \mathcal{B}, T : \mathcal{D} \to \mathcal{C}$, there is a unique functor $F : \mathcal{D} \to \mathcal{B} \times \mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{D} \arrow[dl, "R"'] \arrow[d, dashed, "F"'] \arrow[dr, "T"] & \\
\mathcal{B} & \mathcal{B} \times \mathcal{C} \arrow[l, "P"'] \arrow[r, "Q"] & \mathcal{C}
\end{tikzcd}$$

Proposition II.3.4 (Proposition 1 in Mac Lane)

Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be categories. For all objects $c \in \mathcal{C}$ and $b \in \mathcal{B}$, let

$$L_c : \mathcal{B} \to \mathcal{D}, \qquad M_c : \mathcal{C} \to \mathcal{D}$$

be functors such that $M_b(c) = L_c(b)$ for all b and c.

Then there exists a bifunctor $S : \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ with $S(-, c) = L_c$ for all c and $S(b, -) = M_b$ for all b, if and only if, for every pair of arrows $f : b \to b'$, $g : c \to c'$,

$$M_{b'} g \circ L_c f = L_{c'} f \circ M_b g. \tag{1}$$

These equal arrows in $\mathcal{D}$ are then the value $S(f, g)$ of the arrow function S of f and g.

Proof.

Write b, c to mean $1_b, 1_c$ for any objects $b \in \mathcal{B}, c \in \mathcal{C}$.

Then

$$\langle b', g \rangle \circ \langle f, c \rangle = \langle b' \circ f, g \circ c \rangle = \langle f \circ b, c' \circ g \rangle = \langle f, c' \rangle \circ \langle b, g \rangle.$$

Suppose that S exists, then applying S to this gives

$$S(b', g) \circ S(f, c) = S(f, c') \circ S(b, g)$$

which is equivalent to (1) by the definition of S.

Conversely, if (1) holds then this defines S satisfying the necessary conditions, and it can be verified that this is a bifunctor. □

Exercise II.3.1

TODO product categories special cases

Exercise II.3.2

TODO preorders

Exercise II.3.3

If $\{\mathcal{C}_i \mid i \in I\}$ is a family of categories indexed by a set I, describe the product $\mathcal{C} = \prod_i \mathcal{C}_i$, its projections $P_i : \mathcal{C} \to \mathcal{C}_i$, and establish the universal property of these projections.

Solution.

$$\mathrm{Ob}(\mathcal{C}) = \left\{ (c_i)_{i \in I} \mid c_i \in \mathrm{Ob}(\mathcal{C}_i) \right\};$$

$$\mathrm{Ar}(\mathcal{C}) = \left\{ (f_i)_{i \in I} \mid f_i \in \mathrm{Ar}(\mathcal{C}_i) \right\};$$

$$P_j\left((c_i)_{i \in I} \right) = c_j.$$

The universal property of these projections in that, given a category $\mathcal{D}$ and functors $\{S_i \mid i \in I\}$, there exists a unique functor $F : \mathcal{D} \to \mathcal{C}$ such that $F \circ P_i = S_i$ for each $i \in I$.

Exercise II.3.4

TODO describe opposite of Matr

Exercise II.3.5

TODO topological spaces

II.4 Functor categories

Definition II.4.1

Given categories $\mathcal{B}, \mathcal{C}$, the functor category $\mathcal{B}^{\mathcal{C}}$ has objects all functors $T : \mathcal{C} \to \mathcal{B}$ and arrows natural transformations between those functors.

For natural transformations $\sigma : R \dot{\to} S, \tau : S \dot{\to} T$, we define the composite $(\tau \cdot \sigma)c \coloneqq \tau c \circ \sigma c$. This is a "vertical" composition, as demonstrated by the following diagram for any $f : b \to b' \in \mathcal{B}$:

$$
\begin{tikzcd}
Rc \arrow[rr, "Rf"] \arrow[dd, "(\tau \cdot \sigma)c"'] \arrow[dr, "\sigma c"] & & Rc' \arrow[dd, "(\sigma \cdot \tau)c'"'] \arrow[dr, "\sigma c'"] & \\
& Sc \arrow[rr, "Sf"] \arrow[dl, "\tau c"] & & Sc' \arrow[dl, "\tau c'"] \\
Tc \arrow[rr, "Tf"] & & Tc' &
\end{tikzcd}
$$

Each parallelogram commutes (since σ and τ natural), so overall the diagram must commute.

Exercise II.4.1

TODO rings

Exercise II.4.2

Describe $\mathcal{B}^X$, for X a finite set.

Solution. Objects are the functors $X \to \mathcal{B}$; that is, functors that assign to each $x_1, x_2, ..., x_n$ an object of $\mathcal{B}$. That is, the objects are n-tuples of objects of $\mathcal{B}$, where $n = |X|$.

Arrows are natural transformations between such functors; because the only arrows in X are the identities, the naturality contraints just enforce that each natural transformation τ is a function between n-tuples of objects of $\mathcal{B}$.

Exercise II.4.3

TODO $\mathbf{Ab}^{\mathbb{N}}$

Exercise II.4.4

TODO preorders

Exercise II.4.5

If G is a finite group, describe $\mathbf{Fin}^G$, where $\mathbf{Fin}$ is the category of finite sets.

Solution. The objects are all functors $T : G \to \mathbf{Fin}$; because G has a single object $*$, each object of $\mathbf{Fin}^G$ is a finite set X (along with some extra structure).

Therefore $\mathbf{Fin}^G = \mathbf{Fin}$.

Exercise II.4.6

TODO infinite cyclic monoid

Exercise II.4.7

TODO

Exercise II.4.8

TODO

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