MT2025 Analysis I lecture notes

Remaining TODOs: 1

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1. The real numbers

1.1. Field axioms

Definition 1.1.1: A field $(F, +, \times)$ is a set F with two binary operations "sum" $+ : F^2 \to F$ and "product" $\times : F^2 \to F$, such that the 10 field axioms A1-A4, M1-M4, D, Z are satisified.

A1-A4 govern addition:
A1 $\forall a, b \in F,\ a + b = b + a$ (commutativity)
A2 $\forall a, b, c \in F,\ (a + b) + c = a + (b + c)$ (associativity)
A3 $\exists 0 \in F$ s.t. $a + 0 = a\ \forall a \in F$ (additive identity)
A4 $\forall a \in F, \exists -a \in F$ s.t. $a + (-a) = 0$ (additive inverse)

M1-M4 deal with multiplication:

M1 $\forall a, b \in F,\ a \times b = b \times a$ (commutativity)

M2 $\forall a, b, c \in F,\ (a \times b) \times c = a \times (b \times c)$ (associativity)

M3 $\exists 1 \in F$ s.t. $\forall x \in F,\ x \times 1 = x$ (multiplicative identity)

M4 $\forall x \in F \setminus \{0\}, \exists\ x^{-1} \in F$ s.t. $x \times x^{-1} = 1$. (multiplicative inverse)

D $\forall a, b, c \in F, a \times (b + c) = (a \times b) + (a \times c)$ (distributivity)

Z $0 \neq 1$

Remark: Without axiom Z, $\{0\}$ would be a field, which is undesirable.

Proposition 1.1.2: If $a + x = a\ \forall a \in \mathbb{R}$, then $x = 0$, i.e. 0 is the unique additive identity.

Proof: As $a + x = a$, then

$$\begin{aligned}(-a) + a + x &= (-a) + a \\ &= 0 \text{ (by A1 and A4)}\end{aligned}$$

By A2,

$$\begin{aligned}\text{LHS} &= ((-a) + a + x) \\ &= 0 + x \text{ (by A1 and A4)} \\ &= x \text{ (by A1 and A3).}\end{aligned}$$

Hence $x = 0$. □

Remark: Associativity means that unbracketed finite sums and products are unambiguous.

This isn't true for infinite sums and products in general.

Proposition 1.1.3: $a \times 0 = 0$ for all $a \in \mathbb{R}$.

Proof:

$$\begin{aligned} a \times 0 &= a \times (0 + 0) \\ &= (a \times 0) + (a \times 0) \end{aligned}$$

so $a \times 0 = 0$ by additive inverse.

□

Proposition 1.1.4: If $a, b \neq 0$ then $a \times b \neq 0$.

Proof: By contraposition.

Suppose $a \times b = 0$ and $a \neq 0$.

Then

$$a^{-1} \times (a \times b) = a^{-1} \times 0 = 0.$$

By M2, M1, M4, LHS $= 1 \times b = 0$, so by M3, $b = 0$. □

Remark: We write ab to mean $a \times b$.

Remark: We write $a - b$ to mean $a + (-b)$, and $a/b = ab^{-1}$.

Remark: For $a \in R$ and positive integer k, we define $a^1 = a$, and $a^{k+1} = a^k a$.

Remark: For $a \neq 0$, we define $a^0 = 1$.

Remark: For $l > 0$ and $a \neq 0$, we define $a^{-l} = \left(a^l\right)^{-1}$.

1.2. Order axioms

Definition 1.2.1: The order axioms say that there is a subset $\mathbb{P} \subseteq \mathbb{R}$ such that:
P1 If $a \in \mathbb{P}, b \in \mathbb{P}$ then $a + b \in \mathbb{P}$
P2 If $a \in \mathbb{P}, b \in \mathbb{P}$ then $ab \in \mathbb{R}$
P3 For all $a \in \mathbb{R}$, precisely one of $a \in \mathbb{P}, a = 0, -a \in \mathbb{P}$ holds (trichotomy)

We say "a is positive" when $a \in \mathbb{P}$.

As an interval, $\mathbb{P} = (0, \infty)$.

Definition 1.2.2: For $a, b \in \mathbb{R}$, we write $a \leq b \Leftrightarrow b - a \in \mathbb{P} \cup \{0\}$ and $a < b \Leftrightarrow b - a \in \mathbb{P}$.

Proposition 1.2.3: $\leq$ is a total order on $\mathbb{R}$.

Proof:

$\leq$ is reflexive: $a - a = 0 \in \mathbb{P} \cup \{0\}$, so $a \leq a$ for all $a \in \mathbb{R}$.

$\leq$ is antisymmetric: suppose $a \leq b$ and $b \leq a$. Then $b - a \in \mathbb{P} \cup \{0\}$ and $-(b - a) = a - b \in \mathbb{P} \cup \{0\}$. These contradict P3 unless $b - a = 0 \Longrightarrow a = b$.

$\leq$ is transitive: suppose $a \leq b$ and $b \leq c$. Then $b - a, c - b \in \mathbb{P} \cup \{0\}$. If a pair of a, b, c are equal, clearly $a \leq c$. Otherwise, $b - a, c - b \in \mathbb{P}$ so by P1, $c - a = (c - b) + (b - a) \in \mathbb{P}$ so $a \leq c$.

Every pair $a, b \in \mathbb{R}$ is comparable:
- If $a = b$, done
- If $a < b$, done
- By trichotomy, the only other option is $a - b = -(b - a) \in \mathbb{P}$, so $b < a$ (so $b \leq a$).

□

Proposition 1.2.4: Let $a, b, c \in \mathbb{R}$.
(i) If $a \leq b$ then $a + c \leq b + c$
(ii) If $a \leq b$ and $c \geq 0$ then $ac \leq bc$

Proof (i): $(b + c) - (a + c) = b - a \in \mathbb{P} \cup \{0\}$ by assumption. □

Proof (ii):

$bc - ca = c(b - a)$.

If $c = 0$ or $b = a$, then $c(b - a) \in \mathbb{P} \cup \{0\}$.

If $c > 0$ and $b > a$, then $c(b - a) \in \mathbb{P}$ by P2. □

Proposition 1.2.5: $\mathbb{C}$ is not an ordered field, i.e. there is no $\mathbb{P} \subseteq \mathbb{C}$ satisfying axioms P1-3.

Proof: Suppose for the sake of contradiction that there is such a $\mathbb{P}$.

By P3, one of $i \in \mathbb{P}, i = 0, -i \in \mathbb{P}$.

If $i = 0$, then $1 = i^4 = 0^4 = 0$, contradicting axiom Z.

If $i \in \mathbb{P}$, then by P2, $-i = i^3 \in \mathbb{P}$. This contradicts P3.

If $-i \in \mathbb{P}$, we get the same contradiction.

Therefore $\mathbb{C}$ is not an ordered field. □

Definition 1.2.6: We define the maximum function to be

$$\max : \mathbb{R}^2 \to \mathbb{R}; \max(a,b) = \begin{cases} b \text{ if } b \leq a \\ a \text{ if } a < b. \end{cases}$$

This is well-defined by the trichotomy axiom P3.

We define the minimum similarly:

$$\min : \mathbb{R}^2 \to \mathbb{R} = \begin{cases} a \text{ if } a \leq b \\ b \text{ if } b < a. \end{cases}$$

We define the modulus function to be

$$|\cdot| : \mathbb{R} \to \mathbb{R}; |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Lemma 1.2.7: The function $x \mapsto x^2$ is increasing on $[0, \infty)$.

Proof: For $x, y \in [0, \infty)$ s.t. $x \leq y$, $y^2 - x^2 = (y - x)(y + x) \in [0, \infty]$ by P2. This satisfies the definition of $x^2 \leq y^2$, so $x \mapsto x^2$ is increasing on this interval. □

Proposition 1.2.8 (triangle inequality): Given $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

Proof:

$$\begin{aligned} (|a+b|)^2 &= (a+b)^2 \\ &= a^2 + 2ab + b^2 \\ &= |a|^2 + 2ab + |b|^2 \\ &\leq |a|^2 + 2|a||b| + |b|^2 \\ &= (|a| + |b|)^2. \end{aligned}$$

By Lemma 1.2.7, $|a + b| \leq |a| + |b|$.

□

Theorem 1.2.9 (Bernoulli's inequality): Let $n \in \mathbb{N}$ and $x > -1$, then $(1+x)^n \geq 1 + nx$.

Proof: By induction.

When $n = 0$, LHS $= 1 =$ RHS.

Suppose true for n. Then

$$\begin{aligned}(1+x)^{n+1} &= (1+x)(1+x)^n \\ &\geq (1+x)(1+nx) \text{ by ih} \\ &= 1 + (n+1)x + nx^2 \\ &\geq 1 + (n+1)x.\end{aligned}$$

Hence true by induction.

□

1.3. The completeness axiom

Remark: Many fields also satisfy the order axioms (e.g. $\mathbb{Q}, \mathbb{R}$). One way that $\mathbb{Q}$ is different from $\mathbb{R}$ is that the rationals have some "gaps", e.g. $\nexists q \in \mathbb{Q}$ s.t. $q^2 = 2$.

Definition 1.3.1: Let $S \subseteq \mathbb{R}$.

$\alpha \coloneqq \max S$ is the maximum of S if $\alpha \in S$ and $\alpha \geq s$ $\forall s \in S$. The minimum is similarly defined.

u is an upper bound for S if $u \geq s$ $\forall s \in S$; a lower bound is defined similarly.

S is bounded if S has a lower and an upper bound.

Definition 1.3.2: The supremum of a set S, $\sup S$, is the least upper bound of S.

Proposition 1.3.3: For $S \subseteq \mathbb{R}$:
(i) If $\max S$ exists, it is unique.
(ii) If $\sup S$ ("least upper bound") exists, it is unique.
(iii) If $\max S$ exists, $\max S = \sup S$.

Proof:

(i) Suppose a, b are maxima of S. Then $a \geq b$ as a is a max and $b \in S$. Also $b \geq a$ by the same argument. Hence $a = b$.
(ii) A least upper bound is an upper bound which is less than or equal to any upper bound. For a, b two least upper bounds, $a \leq b$ and $b \leq a$ so $a = b$.
(iii) $\max S \leq \sup S$ because $\max S \in S$. But $\max S$ is an upper bound itself, so $\sup S \leq \max S$. Therefore $\max S = \sup S$.

□

Definition 1.3.4: The completeness axiom is:
C Every nonempty bounded above $S \subseteq \mathbb{R}$ has a supremum.

Remark: To verify that α is the supremum of a set S involved showing that:
- $\alpha \geq s \ \forall s \in S$
- if $\beta \geq s \ \forall s \in S$ then $\beta \geq \alpha$

Alternatively, the second step can be replaced with showing that

$$\forall \varepsilon > 0, \ \exists s \in S \text{ s.t. } \alpha - \varepsilon \leq s \leq \alpha.$$

TODO approximation property

Remark: There are many equivalent statements of the completeness axioms.

Definition 1.3.5: The infimum of a set S is the greatest lower bound of S. We don't need any addition axioms to show theexistance of an infimum equivalent to the supremum, because for $S \neq \varnothing$ we can define

$$\inf S := -\sup\{-s \mid s \in S\}.$$

Remark: The reals are essentially unique in satisfying the field, order and completeness axioms.

Theorem 1.3.6: There is a unique $\alpha > 0$ s.t. $\alpha^2 = 2$.

Proof: Let $S = \{x \in \mathbb{R} \mid x^2 \leq 2\}$.

Note that $1 \in S$ and that 2 bounds S from above. We set $\alpha = \sup S$, which exists by the completeness axiom.

By trichotomy, $\alpha^2 = 2$ or $\alpha^2 > 2$ or $\alpha^2 < 2$.

Suppose for the sake of contradiction that $\alpha^2 > 2$. Pick $0 < h < \frac{\alpha^2 - 2}{2\alpha}$, then $(\alpha - h)^2 > 2$. Then no element of S lies in $(\alpha - h, \alpha)$. This contradicts the approximation property.

Suppose for the sake of contradiction that $\alpha^2 < 2$. Pick $h < 1$ (so that $h^2 < h$), then

$$0 < h < \min\left\{1, \frac{2 - \alpha^2}{2\alpha + 1}\right\},$$

giving $(\alpha + h)^2 < 2$. Therefore $\alpha + h \in S$, but $\alpha + h > \alpha = \sup S$, contradiction.

By trichotomy, $\alpha^2 = 2$.

To show uniqueness, suppose that $\beta > 0$ and $\beta^2 = 2$. Then $\alpha^2 = \beta^2 = 2$ and

$$0 = \alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta).$$

Since $\alpha + \beta > 0$, it must be that $\alpha - \beta = 0$ so $\alpha = \beta$. $\square$

Remark: We can generalise these arguments to show the uniqueness of nth roots for positive reals.

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